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1999 J. Phys. A: Math. Gen. 32 6409

(http://iopscience.iop.org/0305-4470/32/36/305)

J. Phys. A: Math. Gen. **32** (1999) 6409–6422. Printed in the UK PII: S0305-4470(99)03142-X

Existence of homoclinic tubes in the sine–Gordon equation with Hamiltonian perturbation

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Received 30 March 1999, in final form 18 May 1999

Abstract. We consider the sine–Gordon equation under Hamiltonian perturbation with even periodic boundary conditions. We give an analytic expression for homoclinic orbits and produce a useful representation of the gradient of an important integral of motion. We establish the existence of homoclinic tubes based on Mel'nikov analysis and an implicit function theorem argument.

1. Introduction

Nonlinear Hamiltonian partial differential equations (PDEs) are of extreme physical importance since they describe processes without dissipation of energy. There is a class of the above PDEs, known as 'soliton equations', including the sine–Gordon equation and the cubic nonlinear Schrödinger equation [7], which admit very regular solutions (almost periodic) in time, under spatially periodic boundary conditions. In these situations dissipative perturbations of the soliton equations produce waves which are observed numerically to possess chaotic behaviour [8]. Thus, near-integrable soliton equations provide natural conditions for the mathematical study of chaotic behaviour in infinite-dimensional dynamical systems. However, the study of near-integrable Hamiltonian PDEs is also important because of the associated questions of the persistence of KAM tori and stochastic layers.

As an example, we consider the sine–Gordon equation with Hamiltonian perturbation. We emphasize that the integrable sine–Gordon equation has an interesting phase-space structure possessing exponentially unstable solutions with associated homoclinic orbits. Ercolani *et al* [3] clarified the role of integrable instabilities with regard to both individual solutions and the corresponding geometry of integrable level sets in the phase space of the sine–Gordon equation. For neutrally stable solutions the level set is an N-torus, $0 \leqslant N \leqslant \infty$ and there are generically incommensurate frequencies in the integrable flow for such data, which consists of the quasiperiodic motion on an N-torus. For quasiperiodic flow on a finite-dimensional unstable torus there are a finite number (k) of unstable modes. Associated with each instability is a family of homoclinic orbits which asymptote, as $t \to \pm\infty$, to the unstable quasiperiodic solution. The associated level set in phase space is a whiskered torus which consists of a torus component together with k-dimensional whiskers.

The present paper is devoted to studying the existence of homoclinic tubes asymptotic to the above whiskered torus of the sine–Gordon equation with conservative-type perturbation. Silnikov studied the structure of the neighbourhood of a homoclinic tube to an invariant torus [12]. He considered a diffeomorphism, T, in a region of finite-dimensional space such that there exists an invariant saddle-type torus, τ. The stable and unstable manifolds of τ intersect transversely with a homoclinic torus τ_h. The existence of a countable collection of homoclinic tori, $\tau_{h,i} = T^i \circ \tau_h$, $i = 0, \pm 1, \pm 2, \ldots$ is defined by the homoclinic tube of τ. He proved the existence of chaotic dynamics in the neighbourhood of the homoclinic tube, as Smale did for the transversal homoclinic orbit.

We are interested in homoclinic tubes for this problem, since each invariant tube contains *non-elliptic* KAM tori and a chaotic region. Recent works [2, 6, 13] on KAM theory for near-integrable Hamiltonian PDEs have focused on the persistence of *elliptic* tori. In the 1970s persistence results for whiskered tori, were obtained and the so-called 'partially hyperbolic KAM theory' due to Graff [4] for finite-dimensional Hamiltonian systems was proposed. Since the original KAM theorem applies only in the regions of phase space foliated by tori and not in regions of phase space containing whiskered tori, KAM theory had to be developed to study the persistence of whiskered tori under perturbations. Moreover, the qualitative analysis of Hamiltonian PDEs with whiskered tori is very complicated and here we expose the first part of this analysis based on the persistence results of whiskered tori.

In [11], the sine–Gordon equation subjected to dissipative perturbations is studied. The persistence of homoclinic orbits is established through a Mel'nikov method [9]. We proved the existence of a codimension-four centre manifold, $\mathcal{M}_\varepsilon$, which contains the whiskered torus and codimension-two centre-stable (unstable) manifolds $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$, $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$, respectively, such that $\mathcal{M}_\varepsilon = W^{\mathrm{s}}(\mathcal{M}_\varepsilon) \cap W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$. Let $\mathcal{T}$ be a submanifold in the intersection of $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$, $\mathcal{T} \subset \mathcal{M}_\varepsilon$. We call H_{tube} a transversal homoclinic tube asymptotic to $\mathcal{M}_\varepsilon$ under the flow Φ_ε^t (Φ_ε^t denotes the evolution operator of the perturbed system) if the intersection between $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ is transversal at $\mathcal{T}$ and

$$H_{\mathrm{tube}} = \bigcup\nolimits_{t \in \mathbb{R}} \Phi_\varepsilon^t \circ \mathcal{T}.$$

We give a coordinate expression for the submanifold $\mathcal{T}$ and investigate this intersection through the existence of simple zeros of the distance function between $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$, $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ and the implicit function theorem argument.

We now describe the structure of this paper. In section 2, we formulate the problem. In section 3, we briefly discuss the characteristic properties of the unperturbed system based on its Lax pair formulation. We prove the expression of homoclinic orbits through Hirota's method and explicity compute the gradient of the Floquet discriminant evaluated on the homoclinic orbits. Section 4 is devoted to the analysis of the locally invariant manifolds of the perturbed system based on these coordinate expressions in the subset of phase space. In section 5, we establish the existence of homoclinic tubes based on Mel'nikov analysis and the implicit function theorem.

2. Problem set-up

We consider the following near-integrable infinite-dimensional Hamiltonian system

$$\boldsymbol{u}_t = J[\nabla H_0(\boldsymbol{u}) + \varepsilon \nabla H_1(\boldsymbol{u}, \mu)] \tag{2.1}$$

where

$$\begin{aligned} &H = H_0 + \varepsilon H_1 \\ &H_0(\boldsymbol{u}) = \int_0^L \tfrac{1}{2}(u_t^2 + c^2 u_x^2) + (1 + \cos u)\,\mathrm{d}x \qquad \text{with} \quad \tfrac{1}{4} < c^2 < 1. \end{aligned} \tag{2.2}$$

$\boldsymbol{u} = (u, u_t)$, J is the symplectic 2×2 matrix and $\nabla f = (\frac{\delta f}{\delta u}, \frac{\delta f}{\delta u_t})$, ε is the small perturbation parameter, μ the external parameters of our problem, εH_1 is the Hamiltonian perturbation and

$u(x, t)$ satisfies even periodic boundary conditions in the space variable. The phase or function space is defined as follows:

$$\mathbb{H}^1 = \left\{ \boldsymbol{u} = (u, u_t) : \boldsymbol{u}(-x) = \boldsymbol{u}(x) = \boldsymbol{u}(x + L), \int_0^L |\partial_x \boldsymbol{u}|^2 \mathrm{d}x < \infty \right\} \tag{2.3}$$

denotes the Sobolev space of functions of x which are L-periodic, even and square integrable with a square integrable first derivative on $[0, L)$. One should supply the phase-space $\mathbb{H}^1$ with the symplectic structure given by the 2-form Ω:

$$\Omega(\boldsymbol{u}, \boldsymbol{v}) = \int_0^L (v_t u - u_t v) \,\mathrm{d}x \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^1. \tag{2.4}$$

Let a domain $\mathcal{P} \subset \mathbb{H}^1$ and $\mathcal{P}_1 \subset \mathcal{P}$ be such that for every $\boldsymbol{u}_0 \in \mathcal{P}_1$ there exists a unique solution $\boldsymbol{u}(t) = \Phi_\varepsilon^t(\boldsymbol{u}_0)$, $t \in I$ of (2.1) with the initial condition $\boldsymbol{u}_0 = \boldsymbol{u}(0)$. The set of the mappings

$$\Phi_\varepsilon^t : \mathcal{P}_1 \longrightarrow \mathcal{P} \qquad \boldsymbol{u}_0 \longrightarrow \Phi_\varepsilon^t(\boldsymbol{u}_0) \quad t \in I \subset \mathbb{R}$$

is called the flow of equation (2.1). We refer to the Hamiltonian H_0 with $\varepsilon = 0$ as the unperturbed Hamiltonian. The infinite-dimensional system generated by (2.2) is the integrable sine–Gordon equation

$$u_{tt} - c^2 u_{xx} - \sin u = 0. \tag{2.5}$$

We define the Poisson bracket as

$$\{F, G\} = \int_0^L \left(\frac{\delta F}{\delta u} \frac{\delta G}{\delta u_t} - \frac{\delta F}{\delta u_t} \frac{\delta G}{\delta u} \right) \mathrm{d}x. \tag{2.6}$$

Note that the evolution of any functional F, under the sine–Gordon flow, is governed by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}. \tag{2.7}$$

In this paper, we consider the following Hamiltonian perturbation term H_1:

$$H_1(\boldsymbol{u}) = \int_0^L (au + bV(u)) \,\mathrm{d}x \tag{2.8}$$

with $V(u) = \sum_{J>1} V_J u^{2J+1}$ and equation (2.1) it can be rewritten as

$$u_{tt} - c^2 u_{xx} - \sin u = \varepsilon \left(a_1 + a_2 \frac{\partial \mathrm{V}}{\partial u}(u) \right). \tag{2.9}$$

We study the existence of homoclinic tubes to an invariant whiskered tori for equation (2.9), in the phase-space $\mathbb{H}^1$.

3. Homoclinic orbits and the Floquet discriminant

The sine–Gordon equation under periodic boundary conditions and in the absence of perturbations admits solutions for very special initial data which are homoclinic in time. In this section we briefly discuss the homoclinic structure of the sine–Gordon equation in terms of spectral theory, since this structure is represented by multiple eigenvalues of the associated eigenvalue problem (Lax pair). We find an analytic expression for the homoclinic solutions using Hirota's bilinear form and present the gradient of the Floquet discriminant evaluated on these homoclinic solutions for the computation of the Mel'nikov function.

3.1. Lax pair formulation

An important aspect of the sine–Gordon equation (2.5), directly related to the integrability of this system, is that it arises as the compatibility condition of the following Lax pair of the linear operators:

$$L^{(x)}(\boldsymbol{u},\zeta)\phi = 0 \qquad L^{(t)}(\boldsymbol{u},\zeta)\phi = 0 \tag{3.1}$$

where

$$\begin{aligned} L^{(x)}(\boldsymbol{u},\zeta) &= -\mathrm{i}c\sigma_2\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}}{4}(cu_x+u_t)\sigma_1 - \frac{1}{16\zeta}\exp[\mathrm{i}u\sigma_3] - \zeta I \\ L^{(t)}(\boldsymbol{u},\zeta) &= -\mathrm{i}\sigma_2\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{i}}{4}(cu_x+u_t)\sigma_1 + \frac{1}{16\zeta}\exp[\mathrm{i}u\sigma_3] - \zeta I \end{aligned} \tag{3.2}$$

where σ_i denote the Pauli matrices, I is the identity matrix, $\boldsymbol{u} = (u(x,t), u_t(x,t))$ the potential and $\zeta \in \mathbb{C}$ denotes the spectral parameter.

Since $u \in \mathbb{H}^1$, we consider the first member of the Lax pair (3.1) as an eigenvalue problem with the complex parameter ζ serving the role of an eigenvalue. The spectrum of $L^{(x)}$, denoted by

$$\sigma(L^{(x)}) := \{\zeta \in \mathbb{C} : L^{(x)}\phi = 0, |\phi| < \infty, \forall x\} \tag{3.3}$$

characterizes the solution of the sine–Gordon equation, the potential $\boldsymbol{u}$ also satisfies the sine–Gordon equation and is of spatial period L. Since the coefficients of $L^{(x)}$ are periodic in x, this Sturm–Liouville problem is a Floquet spectral problem. As $u(x,t)$ satisfies the sine–Gordon equation, this Floquet spectrum is invariant in t, a fact which provides a sufficient (countable) number of constants of the motion, to make the equation completely integrable. In this case, the generic level sets of these invariants will be an N-torus ($0 \leqslant N \leqslant \infty$):

$$\cdots \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times \cdots$$

and the solutions, $u(x,t)$, will be almost periodic in time. We do not have the opportunity to display the Floquet spectral theory of (3.1) here, see [1, 3] and references therein. We briefly describe the N-tori of the sine–Gordon equation.

Let us consider the fundamental matrix $M(x, x_0; \boldsymbol{u}, \zeta)$ of $L^{(x)}$ as

$$L^{(x)}(\boldsymbol{u},\zeta)M = 0 \qquad M(x_0, x_0; \boldsymbol{u}, \zeta) = I \tag{3.4}$$

and the Floquet discriminant

$$\Delta(\boldsymbol{u},\zeta) := \mathrm{Tr}\, M(x_0 + L, x_0; \boldsymbol{u}, \zeta).$$

The spectrum of $L^{(x)}$ is given by the following condition:

$$\sigma(L^{(x)}) := \{\zeta \in \mathbb{C} : \Delta(\boldsymbol{u},\zeta) \in \mathbb{R}, |\Delta(\boldsymbol{u},\zeta)| \leqslant 2\}. \tag{3.5}$$

The Floquet discriminant, Δ, is known to be analytic in both its arguments and is invariant along solutions of the sine–Gordon equation. This means that $\Delta(\boldsymbol{u},\zeta)$ satisfies the following Poisson bracket conditions:

$$\{\Delta(\boldsymbol{u},\zeta), \Delta(\boldsymbol{u},\zeta')\} = 0 \qquad \forall \zeta, \zeta' \in \mathbb{C}, \qquad \{\Delta(\boldsymbol{u},\zeta), H(\boldsymbol{u})\} = 0 \qquad \forall \zeta \in \mathbb{C}.$$

There are N-phase solutions, $\phi(\vartheta_1, \vartheta_2, \ldots, \vartheta_N)$, of the Lax pair, with precisely k ($0 \leqslant k \leqslant N$) simple periodic eigenvalues (the simple zeros of $\Delta \pm 2$) in the open first quadrant of the ζ-plane and $2(N-k)$ simple periodic eigenvalues in the imaginary axis. The phase evolution is determined by $L^{(t)}$ with the phases evolving linearly according to

$$\vartheta_j = \kappa_j x + \omega_j t + \vartheta_j^{(0)} \qquad \kappa_j = 2\pi j/L$$

where κ_j, ω_j are determined uniquely by the simple spectrum of $L^{(x)}$ and the phase parameters $\vartheta_j^{(0)} \in [0, 2\pi)$ parametrize an N-torus. For a given N-phase solution, the isospectral set (all sine–Gordon solutions with this spectrum) contains an N-torus, the N-phase solutions explicity parametrize the linear flow on the finite-dimensional tori of this completely integrable Hamiltonian system. If the spectrum of an N-phase solution contains multiple periodic eigenvalues (corresponding to multiple zeros of $\Delta \pm 2$) off the real axis, then exponential instabilities may be present. If these instabilities have a positive growth rate, they correspond to orbits that are asymptotic to the N-phase torus. This corresponds to a whiskered N-torus [8] and the isospectral set includes the homoclinic solutions which approach the torus as $t \to \pm\infty$.

As an example let us compute the spectrum for the spatially and temporally uniform solution $\boldsymbol{u} = O = (0, 0)$. The Floquet discriminant is given by

$$\Delta(\boldsymbol{u}, \zeta) = 2\cos\left[\left(\zeta + \frac{1}{16\zeta}\right)\frac{L}{c}\right]. \tag{3.6}$$

The simple periodic spectrum is given by

$$\zeta = \frac{1}{2}\left(\frac{jc\pi}{L} \pm \sqrt{\left(\frac{jc\pi}{L}\right)^2 - \frac{1}{4}}\right) \qquad j \in \mathbb{Z}. \tag{3.7}$$

Each of these points is a double point embedded in the continuous spectrum and becomes complex if

$$0 \leqslant \left(\frac{2\pi jc}{L}\right)^2 \leqslant 1. \tag{3.8}$$

Condition (3.8) is exactly the same as the condition for linearized instability and the complex double point is given by

$$\zeta_d = \tfrac{1}{4}\exp[\mathrm{i}\beta] \qquad \text{with} \quad \beta = \tan^{-1}\frac{L}{2j\pi c}\sqrt{1 - \left(\frac{2jc\pi}{L}\right)^2}. \tag{3.9}$$

3.2. Homoclinic orbits

Ercolani *et al* [3] proved that homoclinic solutions of the sine–Gordon equation can be obtained by a decomposition of two Bäcklund transformations from the uniform solution $(0, 0)$. Here, we derive the homoclinic solutions following Hirota's method, [5]. Let us consider the unperturbed sine–Gordon equation in the form

$$u_{tt} - u_{xx} + \sin u = 0. \tag{3.10}$$

We assume that the solution of (3.10) has the form

$$u(x, t) = 4\tan^{-1}\left[\frac{g(x, t)}{f(x, t)}\right] \tag{3.11}$$

where g, f are smooth functions. Making use of the identity, $\tan^{-1} X = \frac{1}{2\mathrm{i}}\ln[\frac{1+\mathrm{i}X}{1-\mathrm{i}X}]$, it follows that:

$$\begin{aligned} u &= -2\mathrm{i}\ln\left[\frac{1 + \mathrm{i}g/f}{1 - \mathrm{i}g/f}\right] \\ \sin u &= \frac{4}{f^4(1 + (g/f)^2)^2} fg(f^2 - g^2). \end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) we obtain the following pair of equations:

$$\begin{aligned} &(D_x^2 - D_t^2) f \cdot g = fg \\ &(D_x^2 - D_t^2)(f \cdot f - g \cdot g) = 0 \end{aligned} \tag{3.13}$$

where

$$D_x^2 f \cdot g = f_{xx} g - 2 f_x g_x + f g_{xx}.$$

If we choose

$$f(x,t) = a \cosh(px + \gamma) \qquad g(x,t) = A\cos(Pt + \Gamma) \tag{3.14}$$

one obtains the breather solution

$$u(x,t) = 4\tan^{-1}\left[\frac{A}{a}\cos(Pt+\Gamma)\,\mathrm{sech}(px+\gamma)\right] \tag{3.15}$$

where

$$p^2 + P^2 = 1 \qquad a^2p^2 = A^2P^2 \qquad \gamma, \Gamma \in \mathbb{R}.$$

Using the symmetries

$$x \longrightarrow t \qquad t \longrightarrow xc \qquad u \longrightarrow \pi + u$$

solution (3.15) takes the form

$$u(x,t) = \pi + 4\tan^{-1}\left[\frac{A}{a}\cos(Pcx+\Gamma)\,\mathrm{sech}(pt+\gamma)\right]$$

which satisfies $u_{tt} - c^2 u_{xx} + \sin u = 0$ and translates the point $(\pi, 0)$ at the origin, therefore, we obtain the homoclinic breather solution of the sine–Gordon equation

$$U(x,t) = 4\tan^{-1}(\tan\beta\cos(xc\cos\beta)\,\mathrm{sech}(t\sin\beta)) \tag{3.16}$$

with β defined in (3.9) and

$$\frac{A}{a} = \tan\beta \qquad P = \cos\beta \qquad \Gamma = \gamma = 0 \qquad p = \sin\beta.$$

Note that $U(x,t)$ approaches the uniform solution, $\boldsymbol{u} = (0,0)$, exponentially as $t \to \pm\infty$:

$$U(x,t) = O[\exp(-\sigma|t|)] \qquad \text{as} \quad t \longrightarrow \pm\infty \qquad \sigma := \sqrt{1-(2\pi c/L)^2}. \tag{3.17}$$

3.3. The gradient of the Floquet discriminant

Let us consider the special solution of the unperturbed sine–Gordon equation, $u = (0,0)$. In this case the solutions of system (3.1) are given by

$$f^{\pm}(x,t) = \exp\left[\pm\mathrm{i}\left(\frac{k(\zeta)x}{c} + \lambda(\zeta)t\right)\right]\begin{pmatrix}1\\ \pm\mathrm{i}\end{pmatrix} \tag{3.18}$$

where $k(\zeta) = \zeta + \frac{1}{16\zeta}$, $\lambda(\zeta) = \zeta - \frac{1}{16\zeta}$. For these solutions the spectrum of $L^{(x)}$ has a double complex point at $\zeta_d = \nu = \frac{1}{4}\exp[\mathrm{i}\beta]$.

Let $m(x, x', t, \zeta, O) = [f_{ij}]$ denote the fundamental matrix of (3.1) at (ζ, O), ψ the general solution of Lax pair system and $U(x,t)$ the homoclinic orbit. Let $M(x, x', t, \zeta, U)$ denote the fundamental matrix of (3.1) at $(\zeta, U(x,t))$. Then, for $\zeta^2 \neq \nu^2$, M, m are related by

$$M(x,t,\zeta,U) = G(x,t;\zeta,\nu)m(x,x',t,\zeta,O)G^{-1}(x,t;\zeta,\nu) \tag{3.19}$$

where the G matrix is given by

$$G(x,t;\zeta,\nu) = \begin{pmatrix} -\nu\psi_1\psi_2^{-1} & \zeta \\ -\zeta & \nu\psi_2\psi_1^{-1}\end{pmatrix}. \tag{3.20}$$

Using the fact that the homoclinic orbit (3.16) is generated from O by a composition of two Bäcklund transformations (at $\zeta = \nu$ and $\zeta = -\nu^*$), cf [3]. It follows, after some manipulation, that the fundamental matrix M evaluated on U is given by

$$M(x,t,\zeta,U) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{3.21}$$

where

$$\begin{aligned} M_{11} &:= \frac{\mathrm{i}}{2\sin 2\beta}\left[(\mathrm{e}^{-2\mathrm{i}\beta} f_{22} - \mathrm{e}^{2\mathrm{i}\beta} f_{11}) + \left(f_{12}\frac{\psi_2}{\psi_1} - f_{21}\frac{\psi_1}{\psi_2}\right)\right] \\ M_{12} &:= \frac{-\mathrm{i}}{2\sin 2\beta}\left[\frac{\psi_2}{\psi_1}(f_{11} - f_{22}) - \mathrm{e}^{2\mathrm{i}\beta} f_{12}\left(\frac{\psi_2}{\psi_1}\right)^2 + \mathrm{e}^{-2\mathrm{i}\beta} f_{21}\right] \\ M_{21} &:= \frac{-\mathrm{i}}{2\sin 2\beta}\left[\frac{\psi_1}{\psi_2}(f_{11} - f_{22}) + \mathrm{e}^{2\mathrm{i}\beta} f_{21}\left(\frac{\psi_1}{\psi_2}\right)^2 - \mathrm{e}^{-2\mathrm{i}\beta} f_{12}\right] \\ M_{22} &:= \frac{\mathrm{i}}{2\sin 2\beta}\left[(\mathrm{e}^{-2\mathrm{i}\beta} f_{11} - \mathrm{e}^{2\mathrm{i}\beta} f_{22}) - \left(f_{12}\frac{\psi_2}{\psi_1} - f_{21}\frac{\psi_1}{\psi_2}\right)\right] \end{aligned} \tag{3.22}$$

and ψ_1, ψ_2 denotes any fixed solutions of the linear system (3.1). At this point we derive the expressions of the gradient of the Floquet discriminant, Δ.

Lemma 3.1. *The gradient of Δ with respect to u, u_t evaluated on the homoclinic solutions U is given by*

$$\begin{aligned} \frac{\delta\Delta}{\delta u}(U,\zeta_d) &= -cD_x\left[\frac{\delta\Delta}{\delta u_t}(U)\right] + \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta}}{4c}\{(M_{12}^L M_{11}^2 + (M_{22}^L - 1)M_{11}M_{12} - M_{12}^2) \\ &\quad \times \exp[\mathrm{i}U] + (M_{21}^L M_{22}^2 - M_{12}^L M_{21}^2 + (M_{11}^L - M_{22}^L)M_{21}M_{22})\exp[-\mathrm{i}U]\} \\ \frac{\delta\Delta}{\delta u_t}(U,\zeta_d) &= -\frac{\mathrm{i}}{4c}\left(\frac{1}{2\sin 2\beta}\right)^2\left[E\frac{\psi_2}{\psi_1} + H\frac{\psi_1}{\psi_2}\right] \end{aligned} \tag{3.23}$$

where $M_{ij}^L = M_{ij}(L)$, $M_{ij} = M_{ij}(x)$ (cf (3.22)), $F_\pm = (2c)^{-1}L\cos\beta \pm \beta$ and E, H are functions of $F_\pm$, x and t to be given in (3.33).

Proof. We rewrite the linear operator $L^{(x)}(\boldsymbol{u},\zeta)$ (cf (3.2)) as follows:

$$L^{(x)}(\boldsymbol{u},\zeta) := -cJD_x + \left(A + \frac{B^2}{\zeta} - \zeta\right)I \tag{3.24}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A = \frac{\mathrm{i}}{4}(cu_x + u_t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B = \frac{\mathrm{i}}{4}\begin{pmatrix} \mathrm{e}^{\mathrm{i}u/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}u/2} \end{pmatrix}.$$

Let M be the fundamental matrix to the system

$$L^{(x)}(\boldsymbol{u},\zeta)M = 0 \qquad M(x=0) = I \tag{3.25}$$

evaluated on the homoclinic solution U or equivalently

$$D_x M = \frac{1}{c}J\left(\zeta - \left(A + \frac{B^2}{\zeta}\right)\right)M.$$

Variation of $\boldsymbol{u}$ leads to the variational equation for the variation of M at fixed ζ:

$$D_x\delta M = \frac{1}{c}J\left(\zeta - \left(A + \frac{B^2}{\zeta}\right)\right)\delta M - \frac{J}{c}\left(\delta A + \frac{2B\delta B}{\zeta}\right)M$$

$$\delta M(x,\zeta,\boldsymbol{u}) = -\frac{1}{c}M(x)\int_0^x M^{-1}(y)J\left[\frac{\mathrm{i}}{4}[\delta u_t + c\delta u_x]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right.$$
$$\left. - \frac{\mathrm{i}\delta u}{16\zeta}\begin{pmatrix}\mathrm{e}^{\mathrm{i}u} & 0\\ 0 & -\mathrm{e}^{-\mathrm{i}u}\end{pmatrix}\right]M(y)\mu\,\mathrm{d}y$$

$$\delta\Delta(L,\zeta,\boldsymbol{u}) = -\frac{1}{c}\,\mathrm{Tr}\left[M(L)\int_0^L M^{-1}(y)\mathrm{J}\left[\frac{\mathrm{i}}{4}[\delta u_t + c\delta u_x]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right.\right.$$
$$\left.\left. - \frac{\mathrm{i}\delta u}{16\zeta}\begin{pmatrix}\mathrm{e}^{\mathrm{i}u} & 0\\ 0 & -\mathrm{e}^{-\mathrm{i}u}\end{pmatrix}\right]M(y)\,\mathrm{d}y\right].$$

An explicit calculation yields

$$\frac{\delta\Delta}{\delta u}(\boldsymbol{u},\zeta) = \frac{\mathrm{i}}{4}\,\mathrm{Tr}\left[M(L)D_x[M^{-1}(x)\mathcal{I}M(x)] + \frac{1}{4c\zeta}M(L)M^{-1}(x)\mathcal{E}M(x)\right] \tag{3.26a}$$

$$\frac{\delta\Delta}{\delta u_t}(\boldsymbol{u},\zeta) = -\frac{\mathrm{i}}{4c}\,\mathrm{Tr}[M(L)M^{-1}(x)\mathcal{I}M(x)] \tag{3.26b}$$

where

$$\mathcal{I} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \mathcal{E} = \begin{pmatrix}0 & \mathrm{e}^{-\mathrm{i}u}\\ \mathrm{e}^{\mathrm{i}u} & 0\end{pmatrix}.$$

Using the fact that the elements of the matrix M are L-periodic functions in x the expression (3.26b) takes the form

$$\frac{\delta\Delta}{\delta u_t}(U(x,t)) = -\frac{\mathrm{i}}{4c}\,\mathrm{Tr}\left[M^{-1}(x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}M(x+L)\right]. \tag{3.27}$$

An explicit representation of M^{-1}:

$$M^{-1}(x,\zeta,u) = \begin{pmatrix}M_{22} & -M_{12}\\ -M_{21} & M_{11}\end{pmatrix}$$

allows us to place (3.27) in the form

$$\frac{\delta\Delta}{\delta u_t}(U(x,t)) = -\frac{\mathrm{i}}{4c}\,[(M_{22}(x)M_{11}(x+L) - M_{11}(x)M_{22}(x+L))$$
$$+ (M_{12}(x)M_{21}(x+L) - M_{21}(x)M_{12}(x+L))]\,. \tag{3.28}$$

Substituting the representations (3.18) and (3.22) into (3.28) we find the expression

$$\frac{\delta\Delta}{\delta u_t} = -\frac{\mathrm{i}}{4c}\left(\frac{1}{2\sin 2\beta}\right)^2\left[\left(4\cos\left(\frac{kL}{c} - 2\beta\right) - 2\cos\left(\frac{kL}{c} + 2\beta\right)\right.\right.$$
$$\left. - \exp\left[\mathrm{i}\left(\frac{kL}{c} + 2\beta\right)\right]\right)\frac{\psi_2}{\psi_1} + \left(2\cos\left(\frac{kL}{c} - 2\beta\right)\right.$$
$$\left.\left. + \exp\left[-\mathrm{i}\left(\frac{kL}{c} - 2\beta\right)\right]\right)\frac{\psi_1}{\psi_2}\right] \tag{3.29}$$

where $\psi = (\psi_1, \psi_2)^{\perp}$ are the Bloch functions for the Lax pair of the sine–Gordon equation. From (3.18) we have

$$\psi_1 = 2\cos\left(\frac{kx}{c} + \lambda t\right) \qquad \psi_2 = -2\sin\left(\frac{kx}{c} + \lambda t\right)$$

where k, λ are evaluated on the double complex point $\zeta_d = \frac{1}{4}\mathrm{e}^{\mathrm{i}\beta}$,

$$k(\zeta_d) = \frac{1}{2}\cos\beta \qquad \lambda(\zeta_d) = \frac{\mathrm{i}}{2}\sin\beta.$$

Thus,

$$\frac{\psi_2}{\psi_1} = \frac{1}{A_1}(-\sinh 2t + \mathrm{i}\sin 2x) \qquad \frac{\psi_1}{\psi_2} = \frac{1}{A_2}(\sinh 2t + \mathrm{i}\sin 2x) \tag{3.30}$$

where

$$x := \frac{\cos\beta}{2c}x \qquad t := \frac{\sin\beta}{2}t$$

and

$$A_1 = 2g(\cos^2 x\cosh^2 t - \sin^2 x\sinh^2 \mathrm{t}) \qquad A_2 = 2(\cos^2 x\sinh^2 t - \sin^2 x\cosh^2 \mathrm{t}). \tag{3.31}$$

Inserting (3.30), (3.31) into (3.29) leads to

$$\frac{\delta\Delta}{\delta u_t}(U,\zeta_d) = -\frac{\mathrm{i}}{4c}\left(\frac{1}{2\sin 2\beta}\right)^2\left[E\frac{\psi_2}{\psi_1} + H\frac{\psi_1}{\psi_2}\right] \tag{3.32}$$

where

$$\begin{aligned} E &= \frac{1}{A_1}[2(\cos F_+ - 2\cos F_-)\sinh 2t + \sin F_-\sin 2x \\ &\qquad + \mathrm{i}(\sinh 2t\sin F_+ - 2(\cos F_+ - 2\cos F_-)\sin 2x)] \\ H &= \frac{1}{A_2}[-(2\sinh 2t\cos F_- + \sin 2x\sin 2F_-) + \mathrm{i}(\sinh 2t\sin F_- - 2\cos F_-\sin 2x)] \end{aligned} \tag{3.33}$$

and

$$\mathrm{F}_\pm = \frac{\cos\beta}{2c}L \pm \beta \qquad \beta = \tan^{-1}\frac{L}{2\pi c}\sqrt{1-\left(\frac{2c\pi}{L}\right)^2}.$$

The periodicity of the fundamental matrix M allows us to rewrite the expression (3.26*a*) as follows:

$$\frac{\delta\Delta}{\delta u}(\boldsymbol{u},\zeta) = \frac{\mathrm{i}}{4}\operatorname{Tr}\left[D_x[M^{-1}(x)\mathcal{I}M(x+L)] + \frac{1}{4c\zeta}M(L)M^{-1}(x)\mathcal{E}M(x)\right]. \tag{3.34}$$

Using (3.26*b*), we obtain an analytic expression for $\frac{\delta\Delta}{\delta u}(U,\zeta_d)$:

$$\begin{aligned} \frac{\delta\Delta}{\delta u}(U,\zeta_d) = -cD_x\left[\frac{\delta\Delta}{\delta u_t}(U)\right] &+ \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta}}{4c}\{(M_{12}^L M_{11}^2 + (M_{22}^L - 1)M_{11}M_{12} - M_{12}^2)\exp[\mathrm{i}U] \\ &+ (M_{21}^L M_{22}^2 - M_{12}^L M_{21}^2 + (M_{11}^L - M_{22}^L)M_{21}M_{22})\exp[-\mathrm{i}U]\} \end{aligned} \tag{3.35}$$

where

$$\exp[\mathrm{i}U] = \frac{1 + \mathrm{i}\tan\beta\cos(xc\cos\beta)\,\mathrm{sech}(t\sin\beta)}{1 - \mathrm{i}\tan\beta\cos(xc\cos\beta)\,\mathrm{sech}(t\sin\beta)}$$

and $M_{ij}^L = M_{ij}(L)$, $M_{ij} = M_{ij}(x)$ (cf (3.22)). □

4. Invariant manifolds

Before addressing the existence of the homoclinic tube in section 5, here we discuss the coordinates of locally invariant manifolds of the perturbed system (2.9).

There exists a two-dimensional subspace $\tilde{\mathcal{X}}$

$$\tilde{\mathcal{X}} = \{u \in \mathbb{H}^1 : u_x = 0\}$$

such that the set $\Pi = \tilde{\mathcal{X}} \cap \mathbb{H}^1$ is invariant under the flow of the perturbed system. The manifold Π is real symplectic with a nondegenerate 2-form, Ω_Π, which arises from the restriction of the symplectic structure to Π. For $\varepsilon = 0$, the system (2.5) restricted to Π becomes a one degree-of-freedom completely integrable Hamiltonian system, pendulum system, for which we know the dynamics. Furthermore, the uniform solution $O = (0, 0)$ is a saddle for the reduced pendulum system which lies on the whiskered torus for the full system. On the space $\mathbb{H}^1$ this uniform solution becomes a singular point of the saddle-focus type.

The linearized system of the perturbed sine–Gordon equation in the neighbourhood of the solution $O = (0, 0)$ takes the following form:

$$y_{tt} - c^2 y_{xx} - y = h(y) \tag{4.1}$$

with $h(y) := \varepsilon g(y) + \sin y - y = O(y^3)$ and $g(u) = a_1 + a_2 \partial_u V(u)$.

Remark 4.1. *We obtain the same equation on linearizing the sine–Gordon equation (2.9) near to the homoclinic orbit $U(x, t)$,*

$$y_{tt} - c^2 y_{xx} - y + K(x, t, \varepsilon; t_0) y = \varepsilon F(x, t, t_0) + G(x, t, y, \varepsilon; t_0) \tag{4.2}$$

with

$$K = 1 - \cos U = O(\exp[-2\sigma|t|])$$
$$F = g(U) = O(\exp[-\sigma|t|]) \qquad \sigma := \sqrt{1 - (2\pi c/L)^2}$$
$$\begin{aligned} G &= \varepsilon[g(U + y) - g(U)] + \sin U(\cos y - 1) + \cos U(\sin y - y) \\ &= O(\exp[-\sigma|t|]y^2) + O(y^3) \end{aligned}$$

whence as $t \to \pm\infty$ equation (4.2) leads to (4.1), since $U(x, t) \to 0$.

Letting $\varepsilon = 0$, (4.1) becomes

$$y_{tt} - c^2 y_{xx} - \mathrm{y} = 0 \qquad \tfrac{1}{4} < c^2 < 1. \tag{4.3}$$

Substituting $\mathrm{y} = \hat{y}(t) \cos k_j x$ with $k_j = 2\pi j/L$ and j an arbitrary integer, we obtain from (4.3):

$$\hat{y}'' - (1 - c^2 k_j^2)\hat{y} = 0.$$

This shows that the jth mode grows exponentially if $c^2(2\pi j/L)^2 < 1$, $j = 0, 1$. Now using the even periodic boundary condition we see that our problem has two unstable directions, the zeroth mode in the plane Π and for $j = 1$ off the subspace Π, for the uniform solution $(0, 0)$ which lies on the circle $u = \theta \mathrm{e}^{\mathrm{i}\beta}$. If $c^2(2\pi j/L)^2 > 1$, we find an infinite number of centre directions with frequencies $\omega_j = \sqrt{c^2(2\pi j/L)^2 - 1}$, $j = 2, 3, \ldots$.

We consider a region V of the phase-space $\mathbb{H}^1$ as follows:

$$\begin{aligned} V = \Bigg\{ & u \in \mathbb{H}^1 : u = \mathrm{e}^{\mathrm{i}\beta}\left(\theta + \sum_{j=0}^{1}(r_{\mathrm{s}_j} + r_{\mathrm{u}_j})\cos k_j x + \sum_{j=2}^{\infty} r_{\mathrm{c}_j} \cos k_j x\right) \\ & \beta \in (-\pi/2, \pi/2), \theta \in [0, 2\pi], r_{\mathrm{s}_j}, r_{\mathrm{u}_j}, r_{\mathrm{c}_j} \in \mathbb{R} \Bigg\}. \end{aligned} \tag{4.4}$$

In the new variables $(r_{\mathrm{s}}, r_{\mathrm{u}}, r_{\mathrm{c}})$ the linear centre, centre-stable and centre-unstable manifolds have the following forms:

$$\begin{aligned}
\mathcal{M}_{\mathrm{lin}} &= \{u \in V : r_{\mathrm{s}} = r_{\mathrm{u}} = 0\} \\
W^{\mathrm{s}}(\mathcal{M}_{\mathrm{lin}}) &= \{u \in V : r_{\mathrm{u}} = 0\} \\
W^{\mathrm{u}}(\mathcal{M}_{\mathrm{lin}}) &= \{u \in V : r_{\mathrm{s}} = 0\}.
\end{aligned}$$

As we show in [11], the above representation of the linear manifolds perturb into smooth invariant submanifolds, which for $\varepsilon \leqslant \varepsilon_0$ are locally the graphs of the functions from the subset Y into $\mathbb{H}^1 \times P$ with

$$\begin{aligned}
Y = \{(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{u}}, r_{\mathrm{c}}) \in \mathbb{H}^1 : |r_{\mathrm{s}}| < \delta_{\mathrm{s}}, |r_{\mathrm{u}}| < \delta_{\mathrm{u}}, \|r_{\mathrm{c}}\| \\
< \delta_{\mathrm{c}}\beta \in (-\pi/2, \pi/2), \delta_\theta < \theta < 2\pi - \delta_\theta\}
\end{aligned} \tag{4.5}$$

where $\delta_\theta, \delta_{\mathrm{s}}, \delta_{\mathrm{u}}, \delta_{\mathrm{c}}$ are fixed positive constants and the external parameter space P:

$$P = \{\mu = (c, a, b) : \tfrac{1}{4} < c^2 < 1, a_1 \in (0, a_0), a_2 \in (0, b_0), a_0, b_0 \in \mathbb{R}\}. \tag{4.6}$$

The perturbed sine–Gordon equation is written in the new variable as follows [11]:

$$\begin{aligned}
r_{\mathrm{u},t} &= \lambda_{\mathrm{u}}^{\varepsilon} r_{\mathrm{u}} + P_{\mathrm{u}}(r, \delta; \varepsilon) \\
r_{\mathrm{s},t} &= -\lambda_{\mathrm{s}}^{\varepsilon} r_{\mathrm{s}} + P_{\mathrm{s}}(r, \delta; \varepsilon) \\
r_{\mathrm{c},t} &= \mathcal{A} r_{\mathrm{c}} + P_{\mathrm{c}}(r, \delta; \varepsilon)
\end{aligned} \tag{4.7}$$

with $\lambda_{\mathrm{u}}^{\varepsilon}, \lambda_{\mathrm{s}}^{\varepsilon} \in \mathbb{R}$, $P_{\mathrm{u}}, P_{\mathrm{s}}, P_{\mathrm{c}}$ the nonlinear functions, δ is the localization parameter and the operator $\mathcal{A}$ has imaginary eigenvalues. The expressions of the codimension-two locally centre-stable and centre-unstable manifolds are given by:

$$W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon) = \{r \in Y : r_{\mathrm{u}} = h^{\mathrm{u}}(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{c}}; \varepsilon, \mu)\} \tag{4.8a}$$

$$W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon) = \{r \in \mathrm{Y} : r_{\mathrm{s}} = h^{\mathrm{s}}(\theta, \beta, r_{\mathrm{u}}, r_{\mathrm{c}}; \varepsilon, \mu)\} \tag{4.8b}$$

where $h^{\mathrm{u}} = (h^{\mathrm{u}}_0, h^{\mathrm{u}}_1), h^{\mathrm{s}} = (h^{\mathrm{s}}_0, h^{\mathrm{s}}_1)$ are C^n smooth functions with $n > n_0$. As we proved in [11], the expression of the codimension-four centre manifold $\mathcal{M}_\varepsilon$ is given by

$$\mathcal{M}_\varepsilon = \{r \in Y : r_{\mathrm{u}} = h^{\mathrm{cu}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu), r_{\mathrm{s}} = h^{\mathrm{cs}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu)\} \tag{4.9}$$

with $h^{\mathrm{cu}} = (h^{\mathrm{cu}}_0, h^{\mathrm{cu}}_1), h^{\mathrm{cs}} = (h^{\mathrm{cs}}_0, h^{\mathrm{cs}}_1)$ C^n smooth functions and $\mathcal{M}_\varepsilon = W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon) \cap W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ and then

$$\begin{aligned}
h^{\mathrm{cu}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu) &= h^{\mathrm{u}}(\theta, \beta, h^{\mathrm{cs}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu), r_{\mathrm{c}}; \varepsilon, \mu) \\
h^{\mathrm{cs}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu) &= h^{\mathrm{s}}(\theta, \beta, h^{\mathrm{cu}}(\theta, \beta, r_{\mathrm{c}}; \varepsilon, \mu), r_{\mathrm{c}}; \varepsilon, \mu).
\end{aligned} \tag{4.10}$$

5. Existence of the transversal homoclinic tube

In this section, we prove the existence of the transversal homoclinic tube of the perturbed sine–Gordon equation (2.9). The key principle which we use is to consider the smooth representation of invariant manifolds, a Mel'nikov argument and the implicit function theorem. Let us consider a neighbourhood $\mathcal{U}$ of the open subset Y:

$$\begin{aligned}
\mathcal{U} = \{(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{u}}, r_{\mathrm{c}}) \in Y : |r_{\mathrm{s,u}}| < \delta_0, \|r_{\mathrm{c}}\| < \delta_1 < \delta_{\mathrm{c}}, \beta \in (-\pi/2, \pi/2) \\
\delta_\theta < \theta < 2\pi - \delta_\theta\}
\end{aligned} \tag{5.1}$$

and its stable and unstable boundary as follows:

$$\begin{aligned}
\partial\mathcal{U}^{\mathrm{s}} &= \{(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{u}}, r_{\mathrm{c}}) \in \mathcal{U} : |r_{\mathrm{s}}| = \delta_0\} \\
\partial\mathcal{U}^{\mathrm{u}} &= \{(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{u}}, r_{\mathrm{c}}) \in \mathcal{U} : |r_{\mathrm{u}}| = \delta_0\}.
\end{aligned} \tag{5.2}$$

For any fixed t_*, $t_* > 0$, $\Phi_\varepsilon^{t_*} \circ W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ intersects the stable boundary of $\mathcal{U}$, $\partial\mathcal{U}^{\mathrm{s}}$, at the point with the following coordinates:

$$r_{\mathrm{u}} = h^{\mathrm{u}}_{t_*}(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{c}}; \varepsilon, \mu)$$

where $h^{\mathrm{u}}_\tau \in C^n$ for $n > n_0$. Since the unperturbed system admits homoclinic solutions in time, $W^{\mathrm{s}}(\mathcal{M}_0) \equiv W^{\mathrm{u}}(\mathcal{M}_0)$, this means that

$$[\Phi^{t_*} \circ W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_0)] \cap \partial\mathcal{U}^{\mathrm{s}} = [\Phi^{t_*} \circ W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_0)] \cap \partial\mathcal{U}^{\mathrm{u}}$$

we have

$$h^{\mathrm{u}}_{t_*}(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{c}}; 0, \mu) = h^{\mathrm{s}}(\theta, \beta, r_{\mathrm{s}}, r_{\mathrm{c}}; 0, \mu).$$

We define the distance function between the above points in the perturbed case as

$$\mathcal{D}(\theta_0, \beta_0, r_{\mathrm{c},0}; \varepsilon, \mu) = h^{\mathrm{u}}_{t_*}(r_{\mathrm{s}}; \theta_0, \beta_0, r_{\mathrm{c},0}; \varepsilon, \mu) - h^{\mathrm{s}}(r_{\mathrm{c}}; \theta_0, \beta_0, r_{\mathrm{c},0}; \varepsilon, \mu). \tag{5.3}$$

Following the method described in [11], we express the above function through the Mel'nikov function as follows:

$$\mathcal{D}(\theta_0, \beta_0, r_{\mathrm{c},0}, \delta_0; \varepsilon, \mu) = \varepsilon M(\theta_0, \beta_0; \mu) + \mathrm{O}(\varepsilon^2). \tag{5.4}$$

Now define the C^{n-2} function d:

$$d(\theta_0, \beta_0, r_{\mathrm{c},0}, \delta_0; \varepsilon, \mu) := \frac{1}{\varepsilon}\mathcal{D}(\theta_0, \beta_0, r_{\mathrm{c},0}, \delta_0; \varepsilon, \mu) = M(\theta_0, \beta_0; \mu) + \mathrm{O}(\varepsilon). \tag{5.5}$$

The Mel'nikov function is given by

$$M(\theta_0, \beta_0; \mu) = \int_{-\infty}^{\infty} \langle \nabla\Delta(U(x,t)), G(U(x,t))\rangle \mathrm{d}t \tag{5.6}$$

evaluated on the homoclinic orbit $U(x,t)$, $G = (0, a_1 + a_2\partial_u \mathrm{V}(u))^\top$ and $\langle a, b\rangle = \int_0^L ab\,\mathrm{d}x$. In our case, when the Mel'nikov function is built with H_0

$$M_1(\theta_0, \beta_0; \mu) = \int_{-\infty}^{\infty} \{H_0, H_1\}(U(x,t))\,\mathrm{d}t$$

is identically zero, because from equation (2.7) we have

$$M_1 = -\varepsilon^{-1}\int_{-\infty}^{\infty} \frac{\mathrm{d}H}{\mathrm{d}t}(U(t))\mathrm{d}t = -\varepsilon^{-1}\Big[H\Big(\lim_{t\to\infty} U(t)\Big) - H\Big(\lim_{t\to-\infty} U(t)\Big)\Big] = 0. \tag{5.7}$$

One can dispute that H_0 is the proper invariant to measure the distance between the manifolds $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$, since $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ associated with the unstable mode (for $j = 1$, cf section 4) is also a level set of the Floquet discriminant Δ.

The Mel'nikov function, M, is built with the Floquet discriminant and direct calculation shows that $\frac{\delta\Delta}{\delta u_t}$ is a combination of the gradient of H_0 and I where $I = \int_0^L u_t u_x\,\mathrm{d}x$ is another invariant of the sine–Gordon equation, which does not imply that M is identically zero. When the Mel'nikov function is identically zero, a second-order Mel'nikov calculation is needed to measure the splitting distance between $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$, [10].

We can rewrite the Mel'nikov function M as:

$$M(\theta_0, \beta_0; \mu) = \int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U(x,t))\left[a_1 + a_2\frac{\partial \mathrm{V}}{\partial u}(U(x,t))\right]\mathrm{d}x\,\mathrm{d}t \tag{5.8}$$

where the expression for $\frac{\delta\Delta}{\delta u_t}$ is given in (3.32). The Mel'nikov integral is convergent. Because the Hamiltonian perturbation H_1 is an exponentially decreasing expression evaluated on the homoclinic solution $U(x,t)$, satisfying condition (3.17), from (3.32) we obtain

$$\left|\frac{\delta\Delta}{\delta u_t}\right| \leqslant C\mathrm{e}^{\sigma t} \qquad C > 0.$$

Setting $M = 0$ in (5.8), we obtain an algebraic equation for the parameters

$$a_1 - a_2\tilde{\sigma} = 0 \tag{5.9}$$

where

$$\tilde{\sigma} = \tilde{\sigma}(\theta, \beta) = -\left\{ \int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)\frac{\partial V}{\partial u}(U)\,\mathrm{d}x\,\mathrm{d}t \left(\int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)\mathrm{d}x\mathrm{d}t\right)^{-1}\right\}.$$

We denote the surface defined by (5.9) by E_β:

$$E_\beta : \beta_0 = \mathcal{B}(\theta_0; c).$$

We state the main result in the following theorem.

Theorem 5.1. *There exist a region $\hat{\mathcal{R}} = (0, 2\pi) \times P$ for the parameters and a positive constant $\varepsilon_0 > 0$, such that for any $|\varepsilon| < \varepsilon_0$ and $(\theta, \mu) \in \hat{\mathcal{R}}$, there exists a codimension-four transversal homoclinic tube in the open set Y of phase-space $\mathbb{H}^1$ asymptotic to the codimension-four centre manifold $\mathcal{M}_\varepsilon$.*

Proof. There exists a region $\mathcal{R}$ of the surface E_β such that

$$\frac{\partial M}{\partial \beta_0}(\beta_0, \theta_0; \mu) \neq 0$$

and $|\partial M/\partial\beta_0| < l$, $l > 0$

For any $\hat{p}_0 = (\hat{\theta}_0, \hat{\beta}_0, \hat{\mu}_0) \in \mathcal{R}$

$$\frac{\partial}{\partial\beta_0}\mathrm{d}(\hat{\theta}_0, \hat{\beta}_0, 0; 0, \hat{\mu}_0) = \frac{\partial}{\partial\beta_0}M(\hat{\theta}_0, \hat{\beta}_0, 0; 0, \hat{\mu}_0) \neq 0 \tag{5.10}$$

and

$$\left|\frac{\partial M}{\partial\beta_0}(\hat{p}_0)\right| < l.$$

By the implicit function theorem there is a neighbourhood $\hat{\mathcal{R}}$ of $(\hat{\theta}_0, \hat{\mu}_0)$ and a unique C^{n-2} function

$$\beta_0 = \hat{\mathcal{B}}(\theta_0, r_{\mathrm{c},0}; \varepsilon, \mu)$$

defined in $\hat{\mathcal{R}}$ such that

$$\hat{\mathcal{B}}(\theta_0, 0; 0, \mu) = \hat{\beta}_0$$

and

$$d(\theta_0, \hat{\mathcal{B}}(\theta_0, r_{\mathrm{c},0}; \varepsilon, \mu), r_{\mathrm{c},0}; \varepsilon, \mu) = 0.$$

Since d is a C^{n-2} smooth function by (5.10), we obtain

$$\frac{\partial}{\partial\beta_0}d(\theta_0, \hat{\mathcal{B}}(\theta_0, r_{\mathrm{c},0}; \varepsilon, \mu), r_{\mathrm{c},0}; \varepsilon, \mu) \neq 0$$

and

$$\left|\frac{\partial \mathrm{d}}{\partial\beta_0}\right| < m, \qquad m > 0 \qquad \text{for} \quad (\theta_0, r_{\mathrm{c},0}; \varepsilon, \mu) \in \hat{\mathcal{R}}.$$

Then, $W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ intersect transversely at the neighbourhood $\hat{\mathcal{R}}$. Let

$$\mathcal{V} = \bigcup\nolimits_{\hat{p}_0\in\mathcal{R}}\hat{\mathcal{R}}$$

then there exists a unique $C^{n-2}(\mathcal{V}, \mathbb{R})$ function

$$\beta_0 = \mathcal{B}(\theta_0, r_{c,0}; \varepsilon, \mu) \tag{5.11}$$

in such a way

$$\mathcal{B}(\theta_0, 0; 0, \mu) = \mathcal{B}_0(\theta_0; \mu)$$

and $d(\theta_0, \hat{\mathcal{B}}(\theta_0, r_{c,0}; \varepsilon, \mu), r_{c,0}; \varepsilon, \mu) = 0$.

Relation (5.11) defines a codimension-one submanifold $\hat{\mathcal{M}}_\varepsilon$ of $\mathcal{M}_\varepsilon$. The coordinate expression of the transversal intersection $\mathcal{T}$ codimension-four of $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ is given by

$$\begin{aligned} r_{\mathrm{u}} &= h^{\mathrm{cu}}(\theta_0, \mathcal{B}(\theta_0, r_{c,0}; \varepsilon, \mu), r_{c,0}; \varepsilon, \mu) \\ r_{\mathrm{s}} &= h^{\mathrm{cs}}(\theta_0, \mathcal{B}(\theta_0, r_{c,0}; \varepsilon, \mu), r_{c,0}; \varepsilon, \mu) \\ \beta_0 &= \mathcal{B}(\theta_0, r_{c,0}; \varepsilon, \mu) \end{aligned} \tag{5.12}$$

where $h^{\mathrm{cu}} = (h_0^{\mathrm{cu}}, h_1^{\mathrm{cu}})$, $h^{\mathrm{cs}} = (h_0^{\mathrm{cs}}, h_1^{\mathrm{cs}})$. Define the homoclinic tube H_{tube} as follows:

$$H_{\mathrm{tube}} = \bigcup_{t\in\mathbb{R}} \Phi_\varepsilon^t \circ \mathcal{T}. \tag{5.13}$$

Then H_{tube} is the codimension-four transversal homoclinic tube asymptotic to $\mathcal{T} \subset \mathcal{M}_\varepsilon$. □

6. Conclusions

We have shown the existence of homoclinic tubes for the near-integrable infinite-dimensional Hamiltonian system. The sine–Gordon equation which we chose to study is a nontrivial integrable equation and admits homoclinic solutions in time. We found an analytic expression for these solutions using Hirota's method. We investigated the existence of homoclinic tubes through the Mel'nikov analysis and an argument of the implicit function theorem. Since the system under study is a near-integrable Hamiltonian PDE we are also interested in studying the non-elliptic KAM tori and chaotic dynamics inside the homoclinic tube.

Acknowledgments

The author gratefully acknowledges the Commission of the European Communities for the postdoctoral fellowship No ERBFMBICT983236 and Ch Li for stimulating discussions.

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